PMM U.S.S.R.,Vol.53,No.1,pp.97-101,1989
Printed in Great Britain

0021-8928/89 $10.00+0.00

ON THE ASYMPTOTIC METHOD OF "LARGE λ."

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In developing the results in /1/, where an easily realizable method is presented for constructing any number of terms of the series expansion of the solution for one class of mixed axisymmetric problems of elasticity theory by using the method of large λ, a method is described for also constructing any number of terms of such an expansion for another extensive class of integral equations of mixed problems of elasticity theory and mathematical physics. The algorithm results in simple arithmetic recursion relations which enables the domain of application of the large λ method to be extended to its theoretical limits and enables the solution to be constructed with any degree of accuracy. Two problems on the interaction of a stamp with a rectangle are considered as examples, for which certain new results are obtained. The large-λ method was proposed and developed in /2-5/, etc.

1. Solution of the integral equation. Many plane mixed problems of the mechanics of a continuous medium reduce to solving the integral equation /5/

$$\int_{-1}^{1} \varphi(t) k\left(\frac{t-\tau}{\lambda}\right) dt = \pi f(\tau), \quad |\tau| \leqslant 1 \tag{1.1}$$

where λ is a dimensionless geometric parameter, $f(\tau)$ is a known function, and $k(y)$ is the kernel, which can be represented in the form

$$k(y) = -\ln|y| - F(y), \quad F(y) = \sum_{i=0}^{\infty} d_i y^{2i} \tag{1.2}$$

The last series converges absolutely for $|y| < y_0$, therefore, the analogous series for the function $F((t-\tau)/\lambda)$ will converge for $|t| \leqslant 1$, $|\tau| \leqslant 1$ if $\lambda > 2/y_0$.

It was shown in /5/ that if $f'(\tau) \in L_{p[-1,1]}$, $p > {}^3/_4$, then any solution of integral Eq.(1.1) from the class $L_{p[-1,1]}$, $p \geqslant 1$ will also be a solution of the integral equation

$$\varphi(t) = \frac{1}{\pi\sqrt{1-t^2}}\left[P - \int_{-1}^{1} \frac{f'(\tau)\sqrt{1-\tau^2}}{\tau - t} d\tau + \right. \tag{1.3}$$

$$\left. \frac{1}{\pi\lambda} \int_{-1}^{1} \frac{\sqrt{1-\tau^2}}{\tau-t} d\tau \int_{-1}^{1} \varphi(x) F'\left(\frac{x-\tau}{\lambda}\right) dx\right]$$

$$P = \frac{1}{\ln 2\lambda}\left[\int_{-1}^{1} \frac{f(t)\,dt}{\sqrt{1-t^2}} + \frac{1}{\pi}\int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} \int_{-1}^{1} \varphi(x) F\left(\frac{x-t}{\lambda}\right) dx\right]$$

$$\left(P = \int_{-1}^{1} \varphi(t)\,dt\right)$$

We will seek the solution of integreal Eq.(1.3) in the form /5/

$$\varphi(t) = \sum_{n=0}^{\infty} \lambda^{-2n} \varphi_n(t)$$

Then to determine $\varphi_n(t)$ we obtain the recursion relation

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$$\varphi_0(t) = \frac{g(t)}{\pi\sqrt{1-t^2}}, \quad g(t) = P - \int_{-1}^{1} \frac{f'(\tau)\sqrt{1-\tau^2}}{\tau - t}$$

$$\varphi_m(t) = \frac{2}{\pi^2\sqrt{1-t^2}} \sum_{i=1}^{m} id_i \int_{-1}^{1} \frac{\sqrt{1-\tau^2}\,d\tau}{\tau - t} \int_{-1}^{1} \varphi_{m-i}(x)(x-\tau)^{2i-1}\,dx$$

$$(m \geqslant 1)$$

If the binomial $(x-\tau)^{2i-1}$ in the last relationships is represented in the form of a polynomial and the order of integration is changed, we then obtain

$$\varphi_m(t) = \frac{2}{\pi^2\sqrt{1-t^2}} \sum_{i=1}^{m} id_i \sum_{k=0}^{2i-1} c_{ki}\Phi_{m-i}^{2i-k-1} R_k(t) \quad (m \geqslant 1) \tag{1.4}$$

Here

$$R_k(t) = \int_{-1}^{1} \frac{\sqrt{1-\tau^2}\,\tau^k\,d\tau}{\tau - t}, \quad \Phi_m{}^k = \int_{-1}^{1} \varphi_m(x)\,x^k\,dx \tag{1.5}$$

$$c_{ki} = (-1)^k \frac{(2i-1)!}{k!\,(2i-k-1)!}$$

The singular integral $R_k(t)$ is represented in the form of the polynomial /5/

$$R_{2n+1}(t) = \pi \sum_{j=0}^{n+1} s_{n-j}t^{2j}\,(n \geqslant 0), \quad R_{2n+2}(t) = \pi \sum_{j=0}^{n+1} s_{n-j}t^{2j+1}\,(n \geqslant -1) \tag{1.6}$$

$$s_{-1} = -1, \quad s_0 = \frac{1}{2}, \quad s_k = \frac{(2k-1)!!}{(2k+2)!!} \quad (k \geqslant 1)$$

Substituting (1.6) into (1.4) and changing the order of summation, we obtain the representation

$$\varphi_m(t) = \frac{2}{\pi\sqrt{1-t^2}} \left\{ \sum_{j=0}^{m-1} \alpha_{mj}t^{2j+1} + \sum_{j=0}^{m} \beta_{mj}t^{2j} \right\} \quad (m \geqslant 1) \tag{1.7}$$

in which the coefficients α_{mj} and β_{mj} are determined from the simple recursion relations

$$\alpha_{mj} = 2\sum_{i=j+1}^{m-1} id_i \sum_{n=j}^{i-1} c_{2n,\,i}s_{n-j-1} \sum_{p=0}^{m-i-1} \alpha_{m-i,\,p} \frac{(2p+2i-2n-1)!!}{(2p+2i-2n)!!} + \tag{1.8}$$

$$d_m \sum_{n=j}^{m-1} c_{2n,\,m}s_{n-j-1}\Phi_0^{2m-2n-1} \quad (m \geqslant 1,\ 0 \leqslant j \leqslant m-1)$$

$$\beta_{mj} = 2\sum_{\substack{i=j\\ i\neq 0}}^{m-1} id_i \sum_{\substack{n=j-1\\ n\neq -1}}^{i-1} c_{2n+1,\,i}s_{n-j} \sum_{p=0}^{m-i} \beta_{m-i,\,p} \frac{(2p+2i-2n-3)!!}{(2p+2i-2n-2)!!} +$$

$$md_m \sum_{\substack{n=j-1\\ n\neq -1}}^{m-1} c_{2n+1,\,m}s_{n-j}\Phi_0^{2m-2n-2} \quad (m \geqslant 1, 0 \leqslant j \leqslant m)$$

The constants $\Phi_m{}^k\,(m \geqslant 1)$ were calculated by substituting (1.7) into the second relationship (1.5) when changing from (1.4) to (1.7) and (1.8).

The constants $\Phi_0{}^k$, in relationships (1.8) are expressed in terms of the right-hand side of integral Eq.(1.1) by means of the formula

$$\Phi_0{}^k = P\frac{(2k-1)!!}{(2k)!!} - \Phi_*{}^k, \quad \Phi_*{}^0 = 0 \tag{1.9}$$

$$\Phi_*^{2k+1} = -\int_{-1}^{1} f'(\tau)\sqrt{1-\tau^2}\,T_k(\tau)\,d\tau \quad (k \geqslant 0)$$

$$\Phi_*^{2k+2} = -\int_{-1}^{1} f'(\tau)\sqrt{1-\tau^2}\,\tau T_k(\tau)\,d\tau \quad (k \geqslant 0)$$

$$T_k(\tau) = \sum_{n=0}^{k} \frac{(2n-1)!!}{(2n)!!} \tau^{2(k-n)}$$

Taking account of the linear dependence of Φ_0^k on P given by the first relationship in (1.9), we represent the solution of the initial integral Eq.(1.1) in the form

$$\varphi(t) = \frac{1}{\pi\sqrt{1-t^2}} [P\varphi^{(1)}(t) - \varphi^{(2)}(t)] \tag{1.10}$$

$$\varphi^{(1)}(t) = 1 + 2\sum_{m=1}^{\infty} \lambda^{-2m} \sum_{j=0}^{m} \beta_{mj}^1 t^{2j}$$

$$\varphi^{(2)}(t) = \int_{-1}^{1} \frac{f'(\tau)\sqrt{1-\tau^2}}{\tau - t} d\tau + 2\sum_{m=1}^{\infty} \lambda^{-2m} \Big(\sum_{j=0}^{m-1} \alpha_{mj}^2 t^{2j+1} + \sum_{j=0}^{m} \beta_{mj}^2 t^{2j} \Big)$$

The coefficients β_{mj}^1 are found by the recursion relations from (1.8) in which β_{mj} must be replaced by β_{mj}^1

$$\Phi_0^{2k} = (2k-1)!!/(2k)!! \tag{1.11}$$

and the coefficients α_{mj}^2 and β_{mj}^2 are found from the recursion relations (1.8) in which α_{mj} and β_{mj} must be replaced, respectively, by α_{mj}^2 and β_{mj}^2, where $\Phi_0^k = \Phi_*^k$ and Φ_*^k. is determined by the second, third, and fourth relationships in (1.9).

To find the integral characteristic P of the solution of the integral Eq.(1.1) we use the second equality of (1.3), then

$$P = \Big[\ln 2\lambda - \sum_{k=0}^{\infty} \lambda^{-2k} p_k\Big]^{-1} \Big[P_0 - \sum_{k=1}^{\infty} \lambda^{-2k} p_k^*\Big] \tag{1.12}$$

$$P_0 = \int_{-1}^{1} \frac{f(t)\,dt}{\sqrt{1-t^2}}, \quad p_0 = d_0$$

$$p_k = d_k \sum_{m=0}^{k} q_{mk} \frac{(2k-2m-1)!!}{(2k-2m)!!} + 2\Sigma_1$$

$$p_k^* = d_k \sum_{m=0}^{k} q_{mk} \Phi_*^{2(k-m)} + 2\Sigma_2$$

$$\Sigma_s = \sum_{i=0}^{k-1} d_i \sum_{m=0}^{i} q_{mi} \sum_{p=0}^{k-i} \beta_{k-i,\,p}^s \frac{(2p+2i-2m-1)!!}{(2p+2i-2m)!!} \quad (k \geqslant 1)$$

$$q_{mi} = \frac{(2m-1)!!\,(2i)!}{(2m)!!\,(2m)!\,(2i-2m)!}$$

In a number of cases it is useful to know the following integral characteristic also:

$$M = \int_{-1}^{1} t\varphi(t)\,dt \tag{1.13}$$

Substituting the first relationship of (1.10) into (1.13), we obtain

$$M = -\Phi_*^1 - 2\sum_{m=1}^{\infty} \lambda^{-2m} \sum_{j=0}^{m-1} \alpha_{mj}^2 \frac{(2j+1)!!}{(2j+2)!!} \tag{1.14}$$

It is seen that all the desired quantities associated with the solution of integral Eq. (1.1) can be expressed in terms of elementary functions. The constants Φ_*^k and P_0, representable in the form of integrals of $f(t)$, are in the solution. In the case when $f(t)$ is a polynomial, these integrals are taken in explicit form.

We obtain from the convergence conditions for the series (1.2) that the solution of integral Eq.(1.1) by the method described can be obtained for $\lambda > 2/y_0$ where y_0 is the radius of convergence of the series in (1.2).

2. Examples. To illustrate the effectiveness of the method, let us consider the plane problem of the interaction of a stamp with an elastic rectangle. Some of the results obtained here for this problem are also of independent interest. In a Cartesian x, y coordinate system let a rectangle occupy the domain $0 \leqslant y \leqslant h$, $|x| \leqslant b$. A stamp with a flat base is impressed by an amount δ without friction into the face $y = h$ on a segment $|x| \leqslant a$. The conditions for no normal displacements and shear stresses are given on the faces $y = 0$ and $|x| = b$ (problem **A**). Rigid clamping conditions can also be examined on the face $y = 0$ (problem **B**). These and problems of analogous formulation were also considered elsewhere (for instance,

/6-12/, etc.).

The problems formulated reduce to solving the integral equation

$$\int_{-1}^{1} \psi(\tau) k\left(\frac{t-\tau}{\lambda}\right) d\tau = \pi\left(\frac{\mu\delta}{a(1-\nu)} - \frac{Qk_0}{2a\varepsilon}\right) \quad (|t| \leqslant 1) \tag{2.1}$$

$$\lambda = \frac{h}{a}, \quad \varepsilon = \frac{b}{a}, \quad Q = \int_{-a}^{a} q(x)\,dx, \quad k_0 = \frac{1-2\nu}{2(1-\nu)^2}$$

$$k(y) = \frac{\pi}{\varepsilon} \sum_{n=1}^{\infty} K(\beta_n) \cos \beta_n y, \quad \beta_n = \frac{\pi n h}{b} \tag{2.2}$$

$$K(u) = \frac{\operatorname{ch} 2u - 1}{u(\operatorname{sh} 2u + 2u)} \quad \text{(problem A)}$$

$$K(u) = \frac{2\varkappa \operatorname{sh} 2u - 4u}{u(2\varkappa \operatorname{ch} 2u + 1 + \varkappa^2 + 4u^2)}, \quad \varkappa = 3 - 4\nu \quad \text{(problem B)}$$

where $q(x) = \psi(x/a)$ is the contact stress under the stamp, μ is the shear modulus, ν is Poisson's ratio, and Q is the force acting on the stamp.

λ	N	Q^*	q^*			
			$\tau = 0$	0.4	0.8	0.95
1.2	19	4.803	1.905	1.948	2.381	3.919
1.2	20	4.833	1.910	1.953	2.388	3.975
1.2	21	4.808	1.906	1.949	2.382	3.927
1.5	8	4.012	1.560	1.607	2.003	3.346
1.5	9	4.010	1.559	1.606	2.002	3.340
1.5	10	4.011	1.559	1.607	2.002	3.343
2.0	2	3.159	1.210	1.250	1.582	2.687
2.0	3	3.143	1.213	1.254	1.574	2.637
2.0	4	3.147	1.213	1.254	1.577	2.645

The solution of integral Eq.(2.1) is connected with the solution of (1.1) for $f(\tau) = 1$ with the **kernel** (2.2) by the relationship

$$\psi(\tau) = \frac{\mu\delta}{a(1-\nu)}\left(1 - \frac{k_0 Q^*}{2\varepsilon}\right)\varphi(\tau)$$

$$Q = \frac{\mu\delta}{1-\nu} Q^*, \quad Q^* = P\left(1 + \frac{k_0 P}{2\varepsilon}\right)^{-1}$$

(the quantity P is determined by the expression in the parentheses in (1.3)).

The kernel (2.2) can be represented in the form (1.2), where (B_{2i} are Bernoulli numbers)

$$d_0 = \ln \frac{\pi}{\varepsilon} + \sum_{n=1}^{\infty} n^{-1}[1 - \beta_n K(\beta_n)]$$

$$d_i = \frac{(-1)^i B_{2i}}{2i(2i)!}\left(\frac{\pi}{\varepsilon}\right)^{2i} + \frac{(-1)^i \pi}{(2i)!\,\varepsilon} \sum_{n=1}^{\infty} [1 - \beta_n K(\beta_n)]\beta_n^{2i-1} \quad (i = 1, 2, \ldots)$$

Using the results of Sect.1 we obtain for the problems in question

$$\varphi(t) = \frac{P}{\pi\sqrt{1-t^2}}\left[1 + 2\sum_{m=1}^{N} \lambda^{-2m} \sum_{j=0}^{m} \beta_{mj}^1 t^{2j} + O(\lambda^{-2N-2})\right] \tag{2.3}$$

$$P = \pi\left[\ln 2\lambda - \sum_{k=0}^{N} \lambda^{-2k} p_k + O(\lambda^{-2N-2})\right]^{-1}$$

where β_{mj}^1 are evaluated from (1.8) by recursion relations in which β_{mj} must be replaced by β_{mj}^1, and Φ_0^{2k} is taken from (1.11), while p_k are evaluated by formulas from (1.12).

Formulas (2.3) are written, to terms $O(\lambda^{-2N-2})$ and the value of N is selected as a function of the given accuracy.

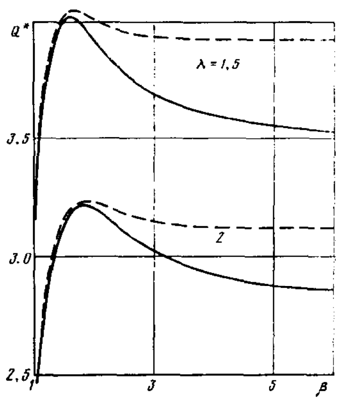
As numerical experiments showed, the convergence of the method (the selection of the value of N) is independent of the parameter ε and improves as λ increases. The solution can here be obtained to any degree of accuracy for $\lambda > 1$. It is important to note that the coefficients of powers of λ^{-1} are sign-variable in the sums from (2.3).

To obtain a given accuracy, 1%, say, we should take $N = 3$ for $\lambda = 2$, $N = 8$ for $\lambda = 1.3$, $N = 17$ for $\lambda = 1.2$, and $N = 26$ for $\lambda = 1.15$, in (2.3).

To demonstrate the convergence of the large-λ method for problem A, values of the quantity Q^* characterizing the stiffness of the rectangle and the magnitudes of the dimensionless contact stresses under the stamp

$$q^*(\tau) = \frac{a(1-\nu)}{\mu\delta} q(\tau a) = \frac{a(1-\nu)}{\mu\delta} \psi(\tau) \quad (|\tau| \leqslant 1)$$

are presented in the table for certain values of the parameters λ, N, τ and $\beta = b/a = 1.5$.



A non-monotonic dependence of the stiffness of the rectangle (the quantity Q^*) on the parameter β is found for a fixed value of λ as a result of the investigations performed. The dependence of Q^* on β for different λ is shown in the figure for problem A (the solid line) and B (the dashed line). It is seen that the rectangle has maximum stiffness for definite values of β in both problems, and the stiffness decreases and tends to a limit value corresponding to problem for a layer as β increases further. It should be emphasized that as $\beta \to \infty$ in problem B the decreases in Q^* proceeds much more rapidly than in problem A.

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Translated by M.D.F.